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Bounded Boolean Powers and Free Product of GMV-Algebras

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Abstract GMV-algebras are a non-commutative generalization of MV-algebras and they describe a non-commutative many-valued Łukasiewicz logic. We introduce a bounded Boolean power of GMV-algebras. Using the bounded Boolean power of MV-algebras, we show that the free product of an MV-algebra and a Boolean algebra is again a commutative MV-algebra, however the free product of MV-algebras in the variety of GMV-algebras is in general non-commutative (Dvurečenskij and Holland, Algebra Univers., (2010), doi:10.1007/S00012-010-0035-X). In addition, we analyze also a topological version of the Boolean power and we show that these constructions are equivalent.

Keywords Boolean power · MV-algebra · GMV-algebra · Boolean space · Free product · Unital ℓ-group

1 Introduction

Theory of quantum structures was inspired by mathematical foundations of quantum mechanics. Today there is a whole hierarchy of quantum structures: Boolean algebras, orthomodular lattices, orthomodular posets, orthoalgebras, D-posets and effect algebras, etc. For a guide through the realm of quantum structures, see the monograph [7].

The classical measurement is performed through a Boolean algebra whereas classical one through one of other quantum structures.

We often combine together classical and quantum experiments. Such measurement settings were described in [3] and they represent an experiment in which a measurement device

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is classical and the measured particles (parameters) are quantum. For such a combination we need a structure that corresponds to the construction of the bounded Boolean power with the given non-classical structure.

An algebraic construction of bounded Boolean powers was studied for many quantum structures, i.e. for orthomodular posets in 1986 by Pták, [14], for difference posets in 1994 by Dvurečenskij and Pulmannová, [6], and for orthoalgebras in 1995 by Foulis and Pták, [10], etc. Moreover, bounded Boolean powers are examples of tensor products of certain types of structures, for difference posets, see [6, 7].

The construction of Boolean powers was originally introduced in the field of topology for rings by Arens and Kaplansky [1] in 1948. Later it was generalized by Foster [8, 9] for arbitrary universal algebras. In the paper we refer to this construction topological.

Pseudo MV-algebras, [11], and, equivalently, generalized MV-algebras, [15], were independently introduced. They represent a non-commutative extension of MV-algebras and they are an algebraic counterpart of Łukasiewicz many-valued logic.

So far bounded Boolean powers were defined only for commutative algebras and in this paper we show that it is also possible to define them for non-commutative algebras, in particular, we study the Boolean power of GMV-algebras.

Thanks to [4], every GMV-algebra is an interval in a unital ℓ-group (= lattice ordered group (written additively), G, with a *strong unit* u, i.e. an element $u \in G^+$ such that given $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$). Using this basic representation, a free product of GMV-algebras was studied in [5] and the free product of MV-algebras in [13]. In [5], it was shown that the free product of MV-algebras in the category of GMV-algebras can be non-commutative. In what follows, using the bounded Boolean power, we show that the free product of an MV-algebra with a Boolean algebra $\mathbf{2}^n$ taken in the category of GMV-algebras is again an MV-algebra.

The paper is organized in the following way. In Sect. 2 we recall some basic definitions we use in the paper. In the next section we show the properties of an algebraic construction of the bounded Boolean power. The last part contains a definition of a topological construction and the analysis on the relation between these two constructions.

2 Preliminaries

According to [11], a *GMV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$, where the derived operation $\odot$ appearing in the axioms (A6) and (A7) is defined by

$$y \odot x = \left(x^- \oplus y^-\right)^\sim.$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;[1]
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

[1] $\odot$ has a higher priority than $\oplus$.

As an example, if u is an arbitrary positive element of a—not necessarily Abelian—ℓ-group G (= lattice ordered group),

$$\Gamma(G,u) := [0,u]$$

and

$$\begin{aligned} x \oplus y &:= (x+y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $(\Gamma(G,u); \oplus, ^-, ^\sim, 0, u)$ is a GMV-algebra [11].

Due to a famous result of Mundici, MV-algebras are intervals in unital Abelian ℓ-groups, [12], whereas GMV-algebras are intervals in ℓ-groups not necessarily Abelian, [4]. We denote by $\mathcal{GMV}$ the variety of GMV-algebras.

A general definition of the free product in an algebraic structure is in [16], and the free product of GMV-algebras was defined in [5]:

Definition 2.1 We say that a GMV-algebra M is the *free product* of a system of GMV-algebras, $\{M_i : i \in I\}$, in the class of all GMV-algebras, if there is a system of GMV-embeddings, $\{\pi_i : i \in I\}$, (= injective homomorphisms of GMV-algebras) $\pi_i : M_i \to M$ such that

(i) the set $\bigcup_{i\in I} \pi_i(M_i)$ generates M,
(ii) for any GMV-algebra N and a system of homomorphisms $\eta_i : M_i \to N$, for $i \in I$, there exists a (necessarily unique) homomorphism $\gamma : M \to N$ such that $\eta_i = \gamma \circ \pi_i$, for $i \in I$.

The free product M of a system of GMV-algebras, $\{M_i : i \in I\}$, in $\mathcal{GMV}$ coincides with the *coproduct* taken in $\mathcal{GMV}$ (we note that the definition of the coproduct is the same as that for the free product only the system of GMV-homomorphisms, $\{\pi_i\}_i$, is not assume more to be a system of injective mappings. Therefore, we have denoted the free product M (if it exists) by $\coprod_{i\in I} M_i$ as is usually denoted the coproduct. If the system is finite, we simply denote the free product $M = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_n$.

We note that in [5], any system of non-trivial GMV-algebras (i.e. $0 \neq 1$) has the free product of GMV-algebras.

Definition 2.2 Let B be a Boolean algebra and let M be a GMV-algebra. Let us define the set

$$M^*[B] = \left\{ f \in B^M : a \neq b \Rightarrow f(a) \wedge f(b) = 0, |f(M)| < \infty, \bigvee_{x\in M} f(x) = 1 \right\}.$$

The set $M^*[B]$ is a so-called *bounded Boolean power of M by B*.

For any $f \in M^*[B]$, we set $M_f := \{x \in M : f(x) \neq 0\}$.

3 Main Results

In the present section, we show that the bounded Boolean power of a GMV-algebra with any Boolean algebra is again a GMV-algebra.

If a Boolean algebra B is trivial, i.e. $0 = 1$, then the bounded Boolean power of arbitrary GMV-algebra M by B is a one-element set containing the function from M to B mapping all elements of M on 0. So it is isomorphic to the one-element GMV-algebra. In what follows, we consider only non-trivial Boolean algebras in the construction of the bounded Boolean power.

For any element $a \in M$ let us define $\hat{a} \in M^*[B]$ by

$$\hat{a}(x) = \begin{cases} 1, & x = a, \\ 0, & x \neq a. \end{cases} \tag{3.1}$$

Theorem 3.1 *Let M be a GMV-algebra and B be a Boolean algebra. Then $(M^*[B]; \oplus, ^-, ^\sim, \hat{0}, \hat{1})$, where for $f, g \in M^*[B]$, $x \in M$:*

$$(f \oplus g)(x) = \bigvee_{\substack{u,v \in M \\ u \oplus v = x}} f(u) \wedge g(v), \tag{3.2}$$

$$\big(f^-\big)(x) = f\big(x^\sim\big) \quad \text{and} \quad \big(f^\sim\big)(x) = f\big(x^-\big), \tag{3.3}$$

is a GMV-algebra.

The proof of Theorem 3.1 is shifted just after Corollary 3.3. First we prove a general result which shows how to calculate operations in $M^*[B]$.

Lemma 3.2 *Let M be a GMV-algebra, B be a Boolean algebra and $M^*[B]$ be the bounded Boolean power of M by B. Let $p(x_1, x_2, \ldots, x_n)$ be a polynomial expression in the language of GMV-algebras. If $f_i \in M^*[B]$, $i = 1, \ldots, n$, then for any $x \in M$*

$$\big[p(f_1, f_2, \ldots, f_n)\big](x) = \bigvee_{\substack{u_1,u_2,\ldots,u_n \in M \\ p(u_1,u_2,\ldots,u_n) = x}} f_1(u_1) \wedge f_2(u_2) \wedge \cdots \wedge f_n(u_n).$$

Proof First we define GMV-algebra polynomial expressions and their lengths:

(i) 0,1 and variables (denoted e.g. $x, y, \ldots$) are GMV-algebra polynomial expressions of length 1.
(ii) If f is a GMV-algebra polynomial of the length n, then f^-, $f^\sim$ are GMV-algebra polynomials of the length $n + 1$.
(iii) If f, g are GMV-algebra polynomials of lengths m, n, respectively, then $f \oplus g$ is a polynomial expression of the length $m + n + 1$.

(For the simplicity of the proof, we do not include the derived GMV-operations as $\odot$, $\wedge$, $\vee$, etc.)

For the length 1 we have:

Case (i): $p(x_1, x_2, \ldots, x_n) = 0$:

The left-hand side of the expression is equal to 0 in $M^*[B]$, i.e. $\hat{0}$.

The right-hand side, $\bigvee_{\substack{u_1,u_2,\ldots,u_n \in M \\ 0=p(u_1,u_2,\ldots,u_n)=x}} f_1(u_1) \wedge f_2(u_2) \wedge \cdots \wedge f_n(u_n)$, is for $x \neq 0$ equal to 0 (as the empty join) and for $x = 0$ it can be rewritten as

$$\begin{aligned} &\bigvee_{u_1,\ldots,u_n \in M} f_1(u_1) \wedge \cdots \wedge f_n(u_n) \\ &\quad = \left(\bigvee_{u_1 \in M} f_1(u_1)\right) \wedge \cdots \wedge \left(\bigvee_{u_n \in M} f_n(u_n)\right) = 1 \wedge \cdots \wedge 1 = 1. \end{aligned}$$

Thus the right-hand side of the expression is also equal to $\hat{0}$.

Case (ii): $p(x_1, \ldots, x_n) = 1$ is proved analogously.

Case (iii): $p(x_1, \ldots, x_n) = x_i$:

For fixed $x \in M$, the left-hand side of the expression is equal to $f_i(x)$ and the right-hand side is

$$\begin{aligned} &\bigvee_{\substack{u_1,u_2,\ldots,u_n \in M \\ p(u_1,u_2,\ldots,u_n)=x}} f_1(u_1) \wedge f_2(u_2) \wedge \cdots \wedge f_n(u_n) \\ &\quad = \bigvee_{\substack{u_1,\ldots,u_n \in M \\ u_i = x}} f_1(u_1) \wedge f_2(u_2) \wedge \cdots \wedge f_n(u_n) \\ &\quad = \left(\bigvee_{u_1 \in M} f_1(u_1)\right) \wedge \cdots \wedge \left(\bigvee_{u_{i-1} \in M} f_{i-1}(u_{i-1})\right) \\ &\qquad \wedge \bigvee_{\substack{u_i \in M \\ u_i = x}} f_i(u_i) \\ &\qquad \wedge \left(\bigvee_{u_{i+1} \in M} f_{i+1}(u_{i+1})\right) \wedge \cdots \wedge \left(\bigvee_{u_n \in M} f_n(u_n)\right) \\ &\quad = 1 \wedge \cdots \wedge 1 \wedge f_i(x) \wedge 1 \wedge \cdots \wedge 1 = f_i(x). \end{aligned}$$

Let us assume that the claim holds for GMV-algebra polynomial expressions with lengths up to n. Let us have an polynomial expression of the length $n + 1$. Then we have three possible situations:

(1) $p(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)^-$, for $g(x_1, \ldots, x_n)$ with length n,
(2) $p(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)^\sim$, for $g(x_1, \ldots, x_n)$ with length n.
(3) $p(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \oplus h(x_1, \ldots, x_n)$, for $g(x_1, \ldots, x_n)$, $h(x_1, \ldots, x_n)$ with lengths s, t, respectively, where $s + t = n$.

Case (1):

$$
\begin{aligned}
\big[p(f_1,\ldots,f_n)\big](x) &= \big(g(f_1,\ldots,f_n)^-\big)(x) = g(f_1,\ldots,f_n)\big(x^\sim\big) \\
&= \bigvee_{\substack{u_1,u_2,\ldots,u_n\in M\\ g(f_1,\ldots,f_n)=x^\sim}} f_1(u_1)\wedge f_2(u_2)\wedge\cdots\wedge f_n(u_n) \\
&= \bigvee_{\substack{u_1,\ldots,u_n\in M\\ g(f_1,\ldots,f_n)^-=(x^\sim)^-=x}} f_1(u_1)\wedge f_2(u_2)\wedge\cdots\wedge f_n(u_n) \\
&= \bigvee_{\substack{u_1,u_2,\ldots,u_n\in M\\ p(u_1,\ldots,u_n)=x}} f_1(u_1)\wedge f_2(u_2)\wedge\cdots\wedge f_n(u_n).
\end{aligned}
$$

Case (2): It can be proved similarly as Case (1).

Case (3):

$$
\begin{aligned}
&\big[p(f_1,f_2,\ldots,f_n)\big](x) = \big[g(f_1,f_2,\ldots,f_n)\oplus h(f_1,f_2,\ldots,f_n)\big](x) \\
&\quad= \bigvee_{\substack{u,v\in M\\ u\oplus v=x}} g(f_1,f_2,\ldots,f_n)(u)\wedge h(f_1,f_2,\ldots,f_n)(v) \\
&\quad= \bigvee_{\substack{u_1,\ldots,u_n\in M\\ v_1,\ldots,v_n\in M\\ u,v\in M\\ g(u_1,\ldots,u_n)=u\\ h(v_1,\ldots,v_n)=v\\ u\oplus v=x}} \big[f_1(u_1)\wedge\cdots\wedge f_n(u_n)\big]\wedge\big[f_1(v_1)\wedge\cdots\wedge f_n(v_n)\big].
\end{aligned}
$$

For any $f\in M^*[B]$, $i=1,\ldots,n$, we have that $f(x)\wedge f(y)=0$, if $x\neq y$. Thus $u_i=v_i$, $i=1,\ldots,n$ and

$$
\begin{aligned}
\big[p(f_1,f_2,\ldots,f_n)\big](x) &= \bigvee_{\substack{u_1,u_2,\ldots,u_n\in M\\ u,v\in M\\ g(u_1,u_2,\ldots,u_n)=u\\ h(u_1,u_2,\ldots,u_n)=v\\ u\oplus v=x}} f_1(u_1)\wedge f_2(u_2)\wedge\cdots\wedge f_n(u_n) \\
&= \bigvee_{\substack{u_1,u_2,\ldots,u_n\in M\\ g(u_1,u_2,\ldots,u_n)\oplus\\ h(u_1,u_2,\ldots,u_n)=x}} f_1(u_1)\wedge f_2(u_2)\wedge\cdots\wedge f_n(u_n) \\
&= \bigvee_{\substack{u_1,u_2,\ldots,u_n\in M\\ p(u_1,u_2,\ldots,u_n)=x}} f_1(u_1)\wedge f_2(u_2)\wedge\cdots\wedge f_n(u_n).
\end{aligned}
$$

□

Corollary 3.3 *Let M be a GMV-algebra, B a Boolean algebra and $M^*[B]$ be the bounded Boolean power of M by B. Then for $f, g \in M^*[B]$*

$$(f \odot g)(x) = \bigvee_{\substack{u,v \in M\\ u \odot v = x}} f(u) \wedge g(v). \tag{3.4}$$

Proof of Theorem 3.1 First we prove that $M^*[B]$ is closed with respect to operations $\oplus$, $^-$ and $^\sim$:

Let $f, g \in M^*[B]$ and let $a, b \in M$. Since $|f(M)| < \infty$ and $|g(M)| < \infty$, we see from (3.2) that $|(f \oplus g)(M)| < \infty$.

We are going to compute

$$\begin{aligned}
\big((f \oplus g)(a)\big) \wedge \big((f \oplus g)(b)\big) &= \left(\bigvee_{\substack{u,v \in M\\ u \oplus v = a}} f(u) \wedge g(v)\right) \wedge \left(\bigvee_{\substack{s,t \in M\\ s \oplus t = b}} f(s) \wedge g(t)\right)\\
&= \bigvee_{\substack{u,v,s,t \in M\\ u \oplus v = a\\ s \oplus t = b}} f(u) \wedge g(v) \wedge f(s) \wedge g(t)\\
&= \bigvee_{\substack{u,v,t \in M\\ u \oplus v = a\\ u \oplus t = b}} f(u) \wedge g(v) \wedge g(t)\\
&= \bigvee_{\substack{u,v \in M\\ u \oplus v = a\\ u \oplus v = b}} f(u) \wedge g(v) = 0, \quad \text{if } a \neq b.
\end{aligned}$$

For any $u, v \in M$ is $u \oplus v$ equal to some $x \in M$, thus we can compute:

$$\begin{aligned}
\bigvee_{x \in M}\left(\bigvee_{\substack{u,v\\ u \oplus v = x}} f(u) \wedge g(v)\right) &= \bigvee_{u,v \in M} f(u) \wedge g(v)\\
&= \left(\bigvee_{u \in M} f(u)\right) \wedge \left(\bigvee_{v \in M} g(v)\right) = 1 \wedge 1 = 1.
\end{aligned}$$

This finishes the proof that $f \oplus g$ is in $M^*[B]$.

Let $f \in M^*[B]$. From properties of GMV-algebras it follows that unary operations $^-$ and $^\sim$ are bijections of M onto M. Thus $a^- = b^-$, iff $a = b$, iff $a^\sim = b^\sim$ and we have that $f^-(a) \wedge f^-(b) = 0 = f^\sim(a) \wedge f^\sim(b)$, whenever $a \neq b$. $|f^-(M)| = |f(M^-)| = |f(M)| < \infty$ and similarly $|f^\sim(M)| = |f(M^\sim)| = |f(M)| < \infty$. (We have denoted $M^- = \{a^- : a \in M\}$ and $M^\sim = \{a^\sim : a \in M\}$. Certainly, $M^- = M = M^\sim$.)

$$\bigvee_{a \in M} f^-(a) = \bigvee_{a \in M} f\big(a^\sim\big) = \bigvee_{a \in M^-} f\big(a^\sim\big) = \bigvee_{a \in M} f\big(a^{-\sim}\big) = \bigvee_{a \in M} f(a) = 1.$$

Similarly we can prove that $\bigvee_{a \in M} f^\sim(a) = 1$. This proves that f^- and $f^\sim$ are from $M^*[B]$.

It remains to check the axioms of GMV-algebras for $M^*[B]$. We prove only the associativity (A1) to illustrate the power of Lemma 3.2, because the proofs of (A2)–(A8) are similar.

(A1) Associativity: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
Let $f, g, h \in M^*[B]$ and $x \in M$. Then

$$\big[f \oplus (g \oplus h)\big](x) = \bigvee_{\substack{u,v,w \in M \\ u \oplus (v \oplus w) = x}} f(u) \wedge g(v) \wedge h(w),$$

$$\big[(f \oplus g) \oplus h)\big](x) = \bigvee_{\substack{u,v,w \in M \\ (u \oplus v) \oplus w = x}} f(u) \wedge g(v) \wedge h(w).$$

Since associativity holds for GMV-algebras, due to Lemma 3.2 the previous expressions are equal for any $x \in M$, i.e. $f \oplus (g \oplus h) = (f \oplus g) \oplus h$. □

Theorem 3.4 *Let M be a GMV-algebra, B be a Boolean algebra and $M^*[B]$ be the bounded Boolean power of M by B. Then the map $\alpha : M \to M^*[B]$, defined by*

$$\alpha(a) = \hat{a}, \quad a \in M, \tag{3.5}$$

is a GMV-embedding of M into $M^*[B]$.

Proof We have

$$\big[\alpha(a \oplus b)\big](x) = [\widehat{a \oplus b}](x) = \begin{cases} 1, & x = a \oplus b \\ 0, & x \neq a \oplus b \end{cases} \quad \text{and}$$

$$\big[\alpha(a) \oplus \alpha(b)\big](x) = [\hat{a} \oplus \hat{b}](x) = \bigvee_{\substack{u,v \in M \\ u \oplus v = x}} \hat{a}(u) \wedge \hat{b}(v) = \begin{cases} 1, & x = a \oplus b, \\ 0, & x \neq a \oplus b. \end{cases}$$

(We have used the following observation. Whenever u differs from a or v differs from b then $\hat{a}(u) \wedge \hat{b}(v) = 0$.) Thus $\alpha(a \oplus b) = \alpha(a) \oplus \alpha(b)$.

$$\big[\alpha\big(a^-\big)\big] = \big[\widehat{a^-}\big](x) = \begin{cases} 1, & x = a^-, \\ 0, & x \neq a^- \end{cases}$$

and

$$\big[\alpha(a)^-\big](x) = \big[\hat{a}^-\big](x) = \hat{a}\big(x^\sim\big) = \begin{cases} 1, & x^\sim = a \\ 0, & x^\sim \neq a \end{cases} = \begin{cases} 1, & x^{\sim -} = x = a^-, \\ 0, & x \neq a^-. \end{cases}$$

Thus $\alpha(a^-) = \alpha(a)^-$.

Analogously we can prove that $\alpha(a^\sim) = \alpha(a)^\sim$. Let $\alpha(a) = \alpha(b)$, i.e. $\hat{a} = \hat{b}$, which is equivalent to $a = b$. This proves that α is injective. □

A Boolean algebra B can be treated as a special case of GMV-algebras in which the operations are defined as follows for $a, b \in B$: $a \oplus b = a \vee b$, $a \odot b = a \wedge b$, $a^- = a^c = a^\sim$, where ^{c} is the complement in B. Moreover, a GMV-algebra M is a Boolean algebra iff $a \oplus a = a$ for any $a \in M$.

Theorem 3.5 *Let M be a GMV-algebra, B be a Boolean algebra and $M^*[B]$ be the bounded Boolean power of M by B. Then the mapping $\beta : B \to M^*[B]$ defined by*

$$\beta(a)(x) = \begin{cases} a, & x = 1, \\ a^c, & x = 0, \qquad a \in B, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.6}$$

is an injective GMV-algebra homomorphism of B into $M^*[B]$.

Proof Trivially, for any $b \in B$, $\beta(b) \in M^*[B]$. Indeed, if $x \neq y$, $x, y \in M$ then $\beta(b)(x) \wedge \beta(b)(y) = 0$, because $\beta(b)$ is non-zero only for $x, y \in \{0, 1\}$ and when $x \neq y$, we get the result $b_1 \wedge b_1^c = 0$. $|\beta(b)(M)| = |\{b, b^c, 0\}| < \infty$. And $\bigvee_{x \in M} \beta(b)(x) = b \vee b^c \vee \bigvee_{x \in M \setminus \{0,1\}} \beta(b)(x) = b \vee b^c \vee 0 = 1$.

$$\beta(b_1 \oplus b_2) = \beta(b_1 \vee b_2) = \begin{cases} b_1 \vee b_2, & x = 1, \\ b_1^c \wedge b_2^c, & x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$[\beta(b_1) \oplus \beta(b_2)](x) = \bigvee_{\substack{u,v \in M \\ u \oplus v = x}} \beta(b_1)(u) \wedge \beta(b_2)(v)$. The element $\beta(b_i)(t)$ is non-zero only for $t = 0$ and $t = 1$, which gives us the following 4 non zero combinations for (u, v), $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. For $x = 0$, the result is equal to $\beta(b_1)(0) \wedge \beta(b_2)(0) = b_1^c \wedge b_2^c$. For $x = 1$, the result is equal to

$$\begin{aligned} &\big(\beta(b_1)(0) \wedge \beta(b_2)(1)\big) \vee \big(\beta(b_1)(1) \wedge \beta(b_2)(0)\big) \vee \big(\beta(b_1)(1) \wedge \beta(b_2)(1)\big) \\ &\quad = \big(b_1^c \wedge b_2\big) \vee \big(b_1 \wedge b_2^c\big) \vee (b_1 \wedge b_2) = \big(b_1^c \wedge b_2\big) \vee b_1 = b_1 \vee b_2, \end{aligned}$$

where we have used distributivity of B. For $x \notin \{0, 1\}$, the result is 0, as the empty join or a join of zero elements. This concludes the proof of $\beta(b_1 \oplus b_2) = \beta(b_1) \oplus \beta(b_2)$.

$$\beta\big(b^-\big) = \beta\big(b^\sim\big) = \beta\big(b^c\big) = \begin{cases} b^c, & x = 1, \\ (b^c)^c = b, & x = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\big[\beta(b)^-\big](x) = \beta(b)\big(x^\sim\big) = \begin{cases} b, & x^\sim = 1 \\ b^c, & x^\sim = 0 \\ 0, & x^\sim \neq 0, 1 \end{cases} = \begin{cases} b, & x = 1^- = 0, \\ b^c, & x = 0^- = 1, \\ 0, & x \neq 0, 1. \end{cases}$$

Thus $\beta(b^-) = \beta(b)^-$. Similarly we can prove that $\beta(b^\sim) = \beta(b)^\sim$.

It remains to prove injectivity of $\beta(b)$. Let $\beta(b_1) = \beta(b_2)$, then $\beta(b_1)(1) = \beta(b_2)(1)$, i.e. $b_1 = b_2$. □

A finite set of elements of a Boolean algebras B, $\{b_i \in B : i = 1, \dots, n\}$, is said to be a *finite resolution of* 1 in B, if $b_i \wedge b_j = 0$, $i \neq j$ and $\bigvee_{i=1}^n b_i = 1$.

Lemma 3.6 *Let M be a GMV-algebra, B be a Boolean algebra and let $M^*[B]$ be the bounded Boolean power of M by B. Let $\{b_i \in B : i = 1, \dots, n\}$ be a finite resolution of* 1 *in B and let $f_i \in M^*[B]$, $(i = 1, \dots, n)$ be given. Then $f := \bigvee_{i=1}^n (f_i \wedge b_i)$ is an element of $M^*[B]$ where*

$$f(x) = \bigvee_{i=1}^{n} \left[f_i(x) \wedge b_i \right], \quad x \in M. \tag{3.7}$$

Proof Let us denote $f := \bigvee_{i=1}^n (f_i \wedge b_i)$, then it is a mapping from M to B. Let $x, y \in M$, $x \neq y$. Then

$$\begin{aligned} f(x) \wedge f(y) &= \left(\bigvee_{i=1}^{n} f_i(x) \wedge b_i \right) \wedge \left(\bigvee_{i=1}^{n} f_i(y) \wedge f(j) \right) \\ &= \left(\bigvee_{\substack{i,j=1, \\ i \neq j}}^{n} b_i \wedge b_j \wedge f_i(x) \wedge f_j(x) \right) \vee \bigvee_{i=1}^{n} \big(b_i \wedge f_i(x) \wedge f_i(y) \big). \end{aligned}$$

The first join is equal to 0, because $b_i \wedge b_j = 0$, for $i \neq j$. The second join is equal to 0 because $f_i \in M^*[B]$ and $x \neq y \Rightarrow f_i(x) \wedge f_i(y) = 0$, $i = 1, \dots, n$. For each $i = 1, \dots, n$, f_i is non-zero on the finite subset M_{f_i} of M. Then f is certainly non-zero on the subset M_f of $\bigcup_{i=1}^n M_{f_i}$ that is finite.

$$\bigvee_{x \in M} f(x) = \bigvee_{x \in M} \bigvee_{i=1}^{n} \big(f_i(x) \wedge b_i \big) = \bigvee_{i=1}^{n} \left(\left(\bigvee_{x \in M} f_i(x) \right) \wedge b_i \right) = \bigvee_{i=1}^{n} b_i = 1. \qquad \square$$

Let f be an arbitrary element of $M^*[B]$. Then f can be expressed in the form

$$f = \bigvee_{\substack{x \in M \\ f(x) \neq 0}} \big(\alpha(x) \wedge f(x) \big) = \bigvee_{\substack{x \in M \\ f(x) \neq 0}} \big(\hat{x} \wedge f(x) \big).$$

Moreover, such a join is finite.

Lemma 3.7 *Let M be a GMV-algebra, B be the Boolean algebra $\mathbf{2}^n$, $n \geq 1$, and let $M^*[B]$ be the bounded power of M by B. Let $b_1, \dots, b_n \in B$ be all the atoms of B. Then each element f of $M^*[B]$ can be written in the form $\bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$ for some* (*not necessary distinct*) *elements $a_i \in M$, $i = 1, \dots, n$. Moreover, such a representation is unique.*

Proof Let us denote by A_B the set of all atoms of B, i.e. $A_B := \{b_1, \dots, b_n\}$. Then $b_1, \dots, b_n$ they form a resolution of 1 in B.

From Lemma 3.6 we have that for an arbitrary n-element system of elements of M, $\{a_1, \dots, a_n\}$, the element $\bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$ is again from $M^*[B]$.

Let us prove the converse inclusion. Let $f \in M^*[B]$; then the set $M_f = \big\{x \in M : f(x) \neq 0\big\}$ is finite. Let us denote the elements of $M_f = \{x_i : i = 1, \dots, t\}$. (Certainly $t \leq n$, because if t would be greater than n then some distinct x_i and x_j from M_f would share a common atom which is under both $f(x_i)$ and $f(x_j)$ and this contradicts the fact $x_i \neq x_j \Rightarrow f(x_i) \wedge f(x_j) = 0$.)

For each $f(x_i)$, $i = 1, \dots, t$, let us denote by B_i the set of all atoms of B which are under $f(x_i)$. Since $\bigvee A_B = 1$, then $f(x_i) = \bigvee B_i$. Moreover B_i for $i = 1, \dots, t$, form a resolution of A_B. Indeed, $f(x_i) \wedge f(x_j) = 0$, for $i \neq j$ implies $B_i \cap B_j = \emptyset$ and $\bigvee_{i=1}^k f(x_i) = \bigvee_{x \in M} f(x) = 1 = \bigvee A_B$ implies $\bigcup_{i=1}^k B_i = A_B$.

Then $f = \bigvee_{i=1}^t (\hat{x}_i \wedge f(x_i)) = \bigvee_{i=1}^t (\hat{x}_i \wedge (\bigvee_{b \in B_i} b)) = \bigvee_{j=1}^n (\hat{a}_j \wedge b_j)$, where for any atom $b_j \in B$, which is the element of B_k, for some $k = 1, \dots, t$, we have denoted a_j the corresponding value of x_k.

Uniqueness: Let $f = \bigvee_{i=1}^n (\hat{a}_i \wedge b_i) = \bigvee_{j=1}^n (\hat{c}_j \wedge b_j)$, for $a_i, c_j \in M$, $i, j = 1, \dots, n$. Then for any i, $b_i \wedge f(a_i) = b_i = b_i \wedge \bigvee_{j=1}^n (\hat{c}_j(a_i) \wedge b_j) = \hat{c}_i(a_i) \wedge b_i$. Thus $\hat{c}_i(a_i) = 1$, i.e. $c_i = a_i$, for $i = 1, \dots, n$. □

Theorem 3.8 *Let M be a GMV-algebra, B be the Boolean algebra $\mathbf{2}^n$, $n \geq 1$, and $M^*[B]$ be the bounded Boolean power of M by B. Let λ, β be the embeddings of M and B into $M^*[B]$, respectively, defined by* (3.5)–(3.6). *Then the GMV-algebra generated by the set $\{\lambda(a) : a \in M\} \cup \{\beta(b) : b \in B\}$ coincides with $M^*[B]$.*

Proof Let us denote by N the set $\{\lambda(a) : a \in M\} \cup \{\beta(b) : b \in B\}$.

Since λ and β are embeddings into $M^*[B]$, the GMV-algebra generated by N is a subalgebra of $M^*[B]$.

For the converse, from Lemma 3.7 we have that each element of $M^*[B]$ can be expressed in the form $\bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$, for some elements $a_i \in M$ and the atoms of B. It is easily seen that $f(x) = \bigvee_{\substack{1 \leq i \leq n \\ x = a_i}} b_i$.

Let $g := \bigvee_{i=1}^n (\lambda(a_i) \wedge \eta(b_i))$. We show that $f = g$.

From Lemma 3.2 we have that for $x \in M$

$$g(x) = \bigvee_{(u_1 \wedge v_1) \vee \cdots \vee (u_n \wedge v_n) = x} \big(\lambda(a_1)(u_1) \wedge \cdots \wedge \lambda(a_n)(u_n) \wedge \eta(b_1)(v_1) \wedge \cdots \wedge \eta(b_n)(v_n)\big)$$

$$= \bigvee_{(a_1 \wedge v_1) \vee \cdots \vee (a_n \wedge v_n) = x} \eta(b_1)(v_1) \wedge \cdots \wedge \eta(b_n)(v_n).$$

In order to $\eta(b_i)(v_i)$ to be non-zero, each b_i has to be equal either to 1, or to 0. Whenever, $v_i = v_j = 1$ for $i \neq j$, we have intersection of two distinct atoms that is equal to 0. In order to obtain a non-zero outcome, at least one element v_i has be equal to 1. Moreover, if all elements v_i are equal to 0, then $g(0) = b_1^c \wedge \cdots \wedge b_n^c = (b_1 \vee \cdots \vee b_n)^c = 0$. So $g(x) = \bigvee_{\substack{1 \leq i \leq n \\ x = a_i}} b_i = f(x)$. □

We note that the direct product of GMV-algebras is again a GMV-algebra when the operations are defined by coordinates. In particular, the power M^n of a GMV-algebra M is a GMV-algebra for any $n \geq 1$.

Theorem 3.9 *Let M be a GMV-algebra, $B = \mathbf{2}^n$ be a finite Boolean algebra, $n \geq 1$, and $M^*[B]$ be the bounded Boolean power of M by B. Then $M^*[B]$ is isomorphic to the GMV-algebra M^n.*

Proof Let us denote the elements of the set of all atoms in B as b_i, $i = 1, \dots, n$.

Let us define a mapping $h : M^n \to M^*[B]$, $(a_1, a_2, \dots, a_n) \mapsto \bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$.

We prove that h is an isomorphism from M^n onto $M^*[B]$. $h((a_1,\dots,a_n)\oplus(c_1,\dots,c_n)) = h(a_1\oplus c_1,\dots,a_n\oplus c_n) = \bigvee_{i=1}^n(\widehat{a_i\oplus c_i}\wedge b_i)$.

$$\begin{aligned}
&\big[h(a_1,\dots,a_n)\oplus h(c_1,\dots,c_n)\big](x)\\
&\quad=\left(\left[\bigvee_{i=1}^n(\hat{a}_i\wedge b_i)\right]\oplus\left[\bigvee_{j=1}^n(\hat{c}_j\wedge b_j)\right]\right)(x)\\
&\quad=\bigvee_{\substack{u,v\in M\\ u\oplus v=x}}\left(\left[\bigvee_{i=1}^n(\hat{a}_i\wedge b_i)\right](u)\wedge\left[\bigvee_{i=1}^n(\hat{c}_j\wedge b_j)\right](v)\right)\\
&\quad=\bigvee_{\substack{u,v\in M\\ u\oplus v=x}}\left(\bigvee_{i=1}^n\big[\hat{a}_i(u)\wedge\hat{c}_i(v)\wedge b_i\big]\right)=\bigvee_{i=1}^n\left[\left(\bigvee_{\substack{u,v\in M\\ u\oplus v=x}}\big[\hat{a}_i(u)\wedge\hat{c}_i(v)\big]\right)\wedge b_i\right]\\
&\quad=\bigvee_{i=1}^n\big[\widehat{a_i\oplus c_i}(x)\wedge b_i\big]=\left(\bigvee_{i=1}^n[\widehat{a_i\oplus c_i}\wedge b_i]\right)(x).
\end{aligned}$$

This concludes the proof of $h((a_1,\dots,a_n)\oplus(c_1,\dots,c_n)) = h(a_1,\dots,a_n)\oplus h(c_1,\dots,c_n)$. For the operation $^-$ we have:

$$h\big((a_1,\dots,a_n)^-\big)=h\big(a_1^-,\dots,a_n^-\big)=\bigvee_{i=1}^n\big(\widehat{a_i^-}\wedge b_i\big).$$

$$\begin{aligned}
\big[h(a_1,\dots,a_n)^-\big](x)&=h(a_1,\dots,a_n)\big[x^\sim\big]=\left[\bigvee_{i=1}^n(\hat{a}_i\wedge b_i)\right]\big(x^\sim\big)\\
&=\bigvee_{i=1}^n\big(\hat{a}_i\big(x^\sim\big)\wedge b_i\big)=\bigvee_{i=1}^n\big(\widehat{a_i^-}(x)\wedge b_i\big)=\left[\bigvee_{i=1}^n\big(\widehat{a_i^-}\wedge b_i\big)\right](x).
\end{aligned}$$

Indeed, for any $i=1,\dots,n$,

$$\hat{a}_i\big(x^\sim\big)=\begin{cases}1, & x^\sim=a_i\\ 0, & x^\sim\neq a_i\end{cases}=\begin{cases}1, & x^{\sim-}=x=a_i^-\\ 0, & x\neq a_i^-\end{cases}=\widehat{a_i^-}(x).$$

Similarly we can prove that $h((a_1,\dots,a_n)^\sim)=h(a_1,\dots,a_n)^\sim$.

$$h(0,\dots,0)(x)=\bigvee_{i=1}^n\hat{0}(x)\wedge b_i=\begin{cases}\bigvee_{i=1}^n b_i=1, & x=0\\ 0, & x\neq 0\end{cases}=\hat{0}(x).$$

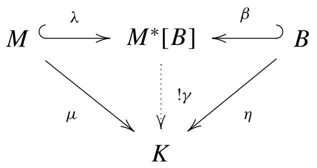
$$h(1,\dots,1)(x)=\bigvee_{i=1}^n\hat{1}(x)\wedge b_i=\begin{cases}\bigvee_{i=1}^n b_i=1, & x=1,\\ 0, & x\neq 1\end{cases}=\hat{1}(x).$$

If $h(a_1,\dots,a_n)=h(c_1,\dots,c_n)$, then certainly $a_i=c_i$, for $i=1,\dots,n$ and $(a_1,\dots,a_n)=(c_1,\dots,c_n)$. This proves the injectivity of h.

From Lemma 3.7 we have that each element $f \in M^*[B]$ can be written in the form $\bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$, for some elements $a_i \in M$, $i = 1, \ldots, n$. Then certainly $h(a_1, \ldots, a_n) = \bigvee_{i=1}^n (\hat{a}_i \wedge b_i) = f$ and thus h is surjective. $\square$

Theorem 3.10 *Let M be a GMV-algebra, $B = \mathbf{2}^n$ be a finite Boolean algebra, $n \geq 1$ and let $M^*[B]$ be the Boolean power of M by B. Then $M^*[B]$ is isomorphic to the free product of B and M taken in the category $\mathcal{GMV}$, i.e. $M^*[B] \cong B \sqcup M \cong M^n$.*

Proof By Lemma 3.7, any element $f \in M^*[B]$ can be uniquely written in the form $f = \bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$, where $\{b_1, \ldots, b_n\}$ is the set of atoms of B. Let K be any GMV-algebra and let $\eta : B \to K$ and $\mu : M \to K$ be GMV-algebra homomorphisms, see the diagram:



Let us define a mapping $\gamma : M^*[B] \to K$ by $\gamma(f) = \bigvee_{i=1}^n [\mu(a_i) \wedge \eta(b_i)]$ whenever $f = \bigvee_{i=1}^n (\hat{a}_i \wedge b_i) \in M^*[B]$. Since K is closed under $\wedge$ and $\vee$, $\gamma(f)$ is an element of K, and due to Lemma 3.7, γ is a well-defined mapping.

Claim 1 *γ is a GMV-algebra homomorphism.*

If $f = \bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$, then $f^- = \bigvee_{i=1}^n (\widehat{a_i^-} \wedge b_i)$. So that $\gamma(f^-) = \bigvee_{i=1}^n [\mu(a_i^-) \wedge \eta(b_i)] = \bigvee_{i=1}^n [\mu(a_i)^- \wedge \eta(b_i)]$.

$$\begin{aligned}
\gamma(f)^- &= \left(\bigvee_{i=1}^n \big(\mu(a_i) \wedge \eta(b_i)\big) \right)^- = \bigwedge_{i=1}^n \big(\mu(a_i)^- \vee \eta(b_i)^-\big) \\
&= \bigwedge_{i=1}^n \left(\mu(a_i)^- \vee \bigvee_{j \neq i} \eta(b_j) \right) = \bigwedge_{i=1}^n \mu(a_i)^- \vee \bigvee_{i=1}^n \big(\mu(a_i)^- \wedge \eta(b_i)\big) \\
&= \bigvee_{i=1}^n \big(\mu(a_i)^- \wedge \eta(b_i)\big)
\end{aligned}$$

because

$$\bigwedge_{i=1}^n \mu(a_i)^- = \left(\bigwedge_{i=1}^n \mu(a_i)^- \right) \wedge \big(\eta(b_1) \vee \cdots \vee \eta(b_n)\big) \leq \bigvee_{i=1}^n \big(\mu(a_i)^- \wedge \eta(b_i)\big).$$

Thus $\gamma(f^-) = \gamma(f)^-$.

In an analogous way, we can prove that $\gamma(f^\sim) = \gamma(f)^\sim$.

Let $f = \bigvee_{i=1}^n (\hat{a}_i \wedge b_i)$ and $g = \bigvee_{i=1}^n (\hat{c}_i \wedge b_i)$. Then

$$\begin{aligned}
\gamma(f \oplus g) &= \gamma \left(\bigvee_{i=1}^n (\widehat{a_i \oplus c_i} \wedge b_i) \right) = \bigvee_{i=1}^n \big(\mu(a_i \oplus c_i) \wedge \eta(b_i)\big) \\
&= \bigvee_{i=1}^n \Big(\big(\mu(a_i) \oplus \mu(c_i)\big) \wedge \eta(b_i)\Big).
\end{aligned}$$

$$\gamma(f)\oplus\gamma(g)=\left(\bigvee_{i=1}^{n}\big(\mu(a_i)\wedge\eta(b_i)\big)\right)\oplus\left(\bigvee_{i=1}^{n}\big(\mu(c_i)\wedge\eta(b_i)\big)\right)$$
$$=\bigvee_{i=1}^{n}\Big(\big(\mu(a_i)\wedge\eta(b_i)\big)\oplus\big(\mu(c_i)\wedge\eta(b_i)\big)\Big)$$
$$\vee\bigvee_{\substack{i,j=1,\\ i\neq j}}^{n}\Big(\big(\mu(a_i)\wedge\eta(b_i)\big)\oplus\big(\mu(c_j)\wedge\eta(b_j)\big)\Big).$$

Since $\oplus$ is distributive over $\vee$ and $\wedge$, [11, Prop. 1.15], we have

$$\begin{aligned}&\big(\mu(a_i)\wedge\eta(b_i)\big)\oplus\big(\mu(c_i)\wedge\eta(b_i)\big)\\ &\quad=\Big(\mu(a_i)\oplus\big(\mu(c_i)\wedge\eta(b_i)\big)\Big)\wedge\Big(\eta(b_i)\oplus\big(\mu(c_i)\wedge\eta(b_i)\big)\Big)\\ &\quad=\big(\mu(a_i)\oplus\mu(c_i)\big)\wedge\big(\mu(a_i)\oplus\eta(b_i)\big)\wedge\big(\eta(b_i)\oplus\mu(c_i)\big)\wedge\big(\eta(b_i)\oplus\eta(b_i)\big)\\ &\quad=\big(\mu(a_i)\oplus\mu(c_i)\big)\wedge\big(\mu(a_i)\oplus\eta(b_i)\big)\wedge\big(\eta(b_i)\oplus\mu(c_i)\big)\wedge\eta(b_i)\\ &\quad=\big(\mu(a_i)\oplus\mu(c_i)\big)\wedge\eta(b_i).\end{aligned}$$

Let us compute the expression for $i\neq j$:

For any $i=1,\dots,n$, the element $\eta(b_i)$ is Boolean, i.e. $\eta(b_i)\oplus\eta(b_i)=\eta(b_i)$. For any element $a\in K$ and a Boolean element $b\in K$ we have $a\oplus b=a\vee b=b\oplus a$, [11, Prop. 4.3]. Hence

$$\begin{aligned}&\big(\mu(a_i)\wedge\eta(b_i)\big)\oplus\big(\mu(c_j)\wedge\eta(b_j)\big)\\ &\quad=\big(\mu(a_i)\oplus\mu(c_j)\big)\wedge\big(\mu(a_i)\oplus\eta(b_j)\big)\wedge\big(\eta(b_i)\oplus\mu(c_j)\big)\wedge\big(\eta(b_i)\oplus\eta(b_j)\big)\\ &\quad=\big(\mu(a_i)\oplus\mu(c_j)\big)\wedge\big(\mu(a_i)\vee\eta(b_j)\big)\wedge\big(\eta(b_i)\vee\mu(c_j)\big)\wedge\big(\eta(b_i)\vee\eta(b_j)\big)\\ &\quad=\big[\mu(a_i)\vee\big(\mu(a_i)\oplus\mu(c_j)\big)\wedge\eta(b_j)\big]\wedge\big(\mu(c_j)\vee\eta(b_i)\big)\wedge\big(\eta(b_i)\vee\eta(b_j)\big)\\ &\quad=\big[\mu(a_i)\wedge\mu(c_j)\vee\mu(c_j)\wedge\eta(b_j)\vee\mu(a_i)\wedge\eta(b_i)\\ &\qquad\vee\big(\mu(a_i)\oplus\mu(c_j)\big)\wedge\eta(b_i)\wedge\eta(b_j)\big]\wedge\big(\eta(b_i)\vee\eta(b_j)\big)\\ &\quad=\mu(a_i)\wedge\mu(c_j)\wedge\big(\eta(b_i)\vee\eta(b_j)\big)\vee\mu(c_j)\wedge\eta(b_j)\vee\mu(a_i)\wedge\eta(b_i)\\ &\quad=\mu(a_i)\wedge\eta(b_i)\vee\mu(c_j)\wedge\eta(b_j)\le\big(\mu(a_i)\oplus\mu(c_i)\big)\wedge\eta(b_i)\vee\big(\mu(a_j)\oplus\mu(c_j)\big)\wedge\eta(b_j).\end{aligned}$$

Claim 2 *If $\delta: M^*[B]\to K$ is a GMV-algebra homomorphism such that $\delta\circ\beta=\eta$ and $\delta\circ\lambda=\mu$, then $\delta=\gamma$.*

Let $f=\bigvee_{i=1}^{n}(\hat{a}_i\wedge b_i)=\bigvee_{i=1}^{n}[\lambda(a_i)\wedge\beta(b_i)]$. Then $\delta(f)=\delta(\bigvee_{i=1}^{n}[\lambda(a_i)\wedge\beta(b_i)])=\bigvee_{i=1}^{n}[\delta(\lambda(a_i))\wedge\delta(\beta(b_i))]=\bigvee_{i=1}^{n}[\mu(a_i)\wedge\eta(b_i)]=\gamma(f)$.

The final conclusion on the free product follows from Theorem 3.9. □

In [5], it was shown that the free product of two MV-algebras taken in the category $\mathcal{GMV}$ is not necessarily an MV-algebra. Theorem 3.10 shows that in some cases, the free product of MV-algebras is again an MV-algebra, that is, if M is an MV-algebra, then $M\sqcup\mathbf{2}^n\cong M^n$. This isomorphism was establish also in [13, Thm. 4.5] for the variety of MV-algebras.

4 Connection with General Topological Construction

Due [2, Chap. IV, §5], a topological construction of a Boolean power was firstly introduced in 1948 by Arens and Kaplansky [1] for rings and later in 1953 it was generalized for general universal algebras by Foster [8, 9]. The main aim of this part is to show a one-to-one correspondence between the algebraic construction, i.e. the construction introduced in Sect. 2 and the topological one.

For a Boolean algebra B we denote B^* the corresponding *Boolean space*, i.e. a completely disconnected compact Hausdorff topological space whose base of clopen subsets forms a Boolean algebra isomorphic to B). Then we have the following definition of a Boolean power [2, Chap. IV, §5].

Definition 4.1 Let B be a Boolean algebra and A be an algebra of some type $\mathcal{K}$. Then $A[B]_t^*$ denotes the set of all continuous functions from B^* to A, giving A the discrete topology. We say that $A[B]_t^*$ is the *(topological) Boolean power of A by B*, where the operations are defined pointwisely.

Due to [2, Thm. IV.5.4] we have the following properties of topological Boolean powers:

Theorem 4.2 *Let A, A_1, A_2 be algebras of some type $\mathcal{K}$ and let B, B_1, B_2 be Boolean algebras. Then the following properties hold*:

(a) *if B is non-trivial, A can be embedded into $A[B]_t^*$*,
(b) $A[\mathbf{2}]_t^* \cong A$,
(c) $A[B_1 \times B_2]_t^* \cong A[B_1]_t^* \times A[B_2]_t^*$,
(d) $(A_1 \times A_2)[B]_t^* \cong A_1[B]_t^* \times A_2[B]_t^*$.

In what follows, we show that there is a one-to-one correspondence between an algebraic construction and a topological construction of Boolean power; then Theorem 3.9 is an easy consequence of points (b), (c) in the previous theorem.

Due to [2, Lem. IV.5.2], elements of $A[B]_t^*$ can be visualized as follows. For each element $f \in A[B]_t^*$, there exists a finite decomposition of B^* into clopen sets $N_1, \dots, N_k$ such that f is constant on each N_i.

From the construction of the Boolean space B^* corresponding to the Boolean algebra B, we have that B^* is the set of all ultrafilters (= maximal proper filters) in B and there is a one-to-one correspondence between elements of B and clopen (= closed-open) sets of B^* given by $a \in B \mapsto N_a = \{U \in B^* : a \in U\}$, and to each clopen set O there is an element $a_O \in B$ such that $O = N_{a_O}$. Moreover, $N_a \cup N_b = N_{a\vee b}$, $N_a \cap N_b = N_{a\wedge b}$, $N'_a = N_{a^c}$, $N_0 = \emptyset$, $N_1 = B^*$.

Due to the visualization, to each N_i there corresponds an element $d_i \in B$, such that $N_i = N_{d_i}$. Since N_i for $i = 1, \dots, k$ form a decomposition of B^*, we have that $d_i \wedge d_j = 0$, for $i \neq j$ and $\bigvee_{i=1}^k d_i = 1$. For each $i = 1, \dots, k$ let us denote a constant value on N_i by a_i.

Without loss of generalization, we can assume that the values of f on each N_i are distinct. (If there were two N_i, N_j, $i \neq j$ such that $a_i = a_j$, then we can simply form another decomposition of B^* containing as subcomponent the union $N_i \cup N_j = N_{d_i\vee d_j}$ which is also a clopen set. Having a finite number of N_i's, we can repeat the previous step to obtain a desired decomposition of B^*.)

Then we construct an element $f_a \in M[B]^*$ such that to each value $a_i \in M$ (constant value on N_i) we simply assign the corresponding value d_i, i.e. $f_a = \bigvee_{i=1}^k (\hat{a}_i \wedge d_i)$.

For the converse, to each element $g \in M[B]^*$ with $g = \bigvee_{i=1}^m (\hat{a}_i \wedge d_i)$ we assign an element $g_t \in M[B]_t^*$ in the following way: the value a_i is obtained on $N_{g(a_i)} = N_{d_i}$.

It is not very hard to see that $f \in M[B]^* \mapsto f_t \in M[B]_t^* \mapsto (f_t)_a = f$ and $g \in M[B]_t^* \mapsto g_a \in M[B]^* \mapsto (g_a)_t = g$, i.e. there is a one-to-one correspondence between elements of $M[B]^*$ and $M[B]_t^*$. Moreover, we prove that the operations on $M[B]^*$ are in a correspondence with operations on $M[B]_t^*$.

If $\{d_i : i = 1, \dots, n\}$ and $\{e_j : j = 1, \dots, m\}$ are two resolutions of 1, the set $\{d_i \wedge e_j : i = 1, \dots, n,\ j = 1, \dots, m\}$ is again a resolution of 1 which is a refinement of any of the given resolutions. Therefore, we can express elements $f, g \in M[B]^*$ over the common resolution $\{f_1, \dots, f_k\}$ of 1. So $f = \bigvee_{i=1}^k (\hat{a}_i \wedge f_i)$ and $g = \bigvee_{i=1}^k (\hat{c}_i \wedge f_i)$. Thus f_t and g_t are constant on N_{f_i} and they take values a_i and c_i, respectively. $f \oplus g$ is then equal $\bigvee_{i=1}^k (\widehat{a_i \oplus c_i} \wedge f_i)$ and thus $(f \oplus g)_t$ has the value $a_i \oplus c_i$ on N_{f_i}, which corresponds to the pointwise operation of $M[B]_t^*$.

The complement f^- is equal $\bigvee_{i=1}^k (\hat{a}_i^- \wedge f_i)$, i.e. $(f^-)_t$ has the value a_i^- on N_{f_i}, which again corresponds to the pointwise operation on $M[B]_t^*$. A similar situation is for $f^\sim$ and also for the both constants 0 and 1.

For the converse, let f, g be elements of $M[B]_t^*$ expressed in an irreducible form, i.e. f attains values a_i on N_{d_i} for $i = 1, \dots, n$ and g attains values c_j on N_{e_j} for $j = 1, \dots, m$ and $a_i \neq a_j$, $c_i \neq c_j$ for $i \neq j$. Then, for each pair (i, j), the value of f and the value of g on $N_{d_i} \cap N_{e_j} = N_{d_i \wedge e_j} \neq \emptyset$ is equal to a_i, c_j, respectively. $f \oplus g$ is computed pointwisely and the irreducible form is obtained in the way described above. For some value $x = a_i \oplus c_j$, which is attained on $N_{d_i \wedge e_j}$, there exist pairs of $(i_1, j_1), \dots, (i_k, j_k)$ such that $x = a_{i_1} \oplus c_{j_1} = \dots = a_{i_k} \oplus c_{j_k}$. We assume that besides these pairs there is no other pair giving value x. Then the value x is attained on $N_{d_{i_1} \wedge e_{j_1}} \cup \dots \cup N_{d_{i_k} \wedge e_{j_k}} = N_d$, where $d = d_{i_1} \wedge e_{i_1} \vee \dots \vee d_{i_k} \wedge e_{i_k}$. So $(f \oplus g)_a(x) = d = \bigvee_{u \oplus v = x} f_a(u) \wedge g_a(v)$. In a similar fashion we prove that $(f^-)_a(x) = (f_a)(x^\sim)$, $(f^\sim)_a(x) = f_a(x^-)$ and $0_a = \hat{0}$, $1_a = \hat{1}$.

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